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EE 364 - Spring 2026

Week 4
Supplemental Notes

Discrete random experiments

Defn: "Bernoulli trial" (a.k.a. coin flip)

binary outcome: H/T, T/F, 0/1, Success / Fail, ...

w/ $P[\text{success}] = p$.

Defn: A Bernoulli sequence $X_1, X_2, \ldots$ is a sequence of independent and identical Bernoulli trials

Defn: A sequence of outcomes $\{x_k\}$ is a random sample iff the outcomes are independent and identically distributed. (i.i.d.) ★★

★★★ random sample $\longleftrightarrow$ i.i.d.

Ex: 2 coin flips

$$P[H_1 \cap T_2] \overset{\text{indep.}}{=} P[H_1] \cdot P[T_2] \overset{\text{ident.}}{=} P[H] \cdot P[T] \quad (= p \cdot q)$$

Binomial probability "distribution"

$$p + \underbrace{(1-p)}_{q} = 1 \qquad 1 = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k \underbrace{(1-p)}_{q}^{n-k}$$

Idea: $\underbrace{\binom{n}{k}}\; p^{\underbrace{k\,(1-p)}^{n-k}}$

prob of getting k "success" and n-k "fail" (ordered)

\# different ways to arrange n "success" and n-k "fail" (n choose k)

$$P[\overbrace{\#\text{success}}^{X} = k] = \binom{n}{k} p^k (1-p)^{n-k} \quad \boxed{\overset{\Delta}{=} b(n, k, p)}$$

(in any order, in n trials)

Binomial pdf (probability "mass" density function)

$$b(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{if} \quad k \le n \quad \text{and} \quad p \in [0,1].$$

p = "success probability"

= P[Success on one Bernoulli Trial].

$\therefore b(n,k,p) = P[\, k \text{ successes in } n\text{-trials} \,]$.

$$P[X \le k] = P[X=0] + P[X=1] + \dots + P[X=k] = \sum_{j=0}^{k} b(n,k,p)$$

$$\therefore \sum_{k=0}^{n} b(n,k,p) = 1 \quad \text{by binomial theorem.}$$

since: $(p + (1-p))^n = 1^n = 1$.

Ex: $P[\, 2 \text{ heads in } 3 \text{ flips} \,] = P[X=2] = b(3, 2, p)$ — Fair, 0.5

means: exactly 2

$$= \binom{3}{2} (0.5)^2 (0.5)^1 = \frac{3}{8}$$

$$P[\text{ at most } 3 \text{ heads in } n \text{ flips }] = P[X \le 3]$$

$$= \sum_{k=0}^{3} b(n, 3, 0.5)$$

$$= \sum_{k=0}^{3} \binom{n}{k} (0.5)^k (0.5)^{n-k}$$

$$= \frac{1}{2^n} \sum_{k=0}^{3} \binom{n}{k}$$

Ex: $P[\, 3 \text{ heads in } 5 \text{ flips} \,] = b(5,3,\tfrac{1}{2}) = \binom{5}{3} (\tfrac{1}{2})^3 (\tfrac{1}{2})^2 = \boxed{5/16}$

$P[\, 3 \text{ heads in } 6 \text{ flips} \,] = b(6,3,\tfrac{1}{2}) = \binom{6}{3} (\tfrac{1}{2})^3 (\tfrac{1}{2})^3 = \boxed{5/16}$

$P[\, 2 \text{ heads in } 5 \text{ flips} \,] = b(5,2,\tfrac{1}{2}) = \binom{5}{2} (\tfrac{1}{2})^2 (\tfrac{1}{2})^3 = \boxed{5/16}$

Pay \$1 to play, \$3 win. Note: average win $\frac{15}{16} < 1$ — "rational price" = expected return

$3 \cdot \frac{5}{16} + 0 \cdot \frac{11}{16}$

Ex: Flip a coin n-times with $p = P(H)$. Suppose n is <u>even</u>. What is the probability of getting $\frac{n}{2}$ heads in n flips

Answer: $b(n, \frac{n}{2}, p) \approx \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} (4pq)^{n/2}$

$\therefore p = q = \frac{1}{2} \longrightarrow b(n,k,p) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$

$\therefore$ Slow (square-root) decay.

computer: $P[X = n \text{ success in } 2n \text{ flips}] = \cdots = \binom{2n}{n} \frac{1}{2^{2n}}$

$$\lim_{n\to\infty} \binom{2n}{n} \cdot \frac{1}{2^{2n}} = ? \qquad (=0)$$

so many ways of getting # $\approx n$ but not n exactly. $\therefore P[X=n] \to 0$

Pf: $b(n,k,p) = \binom{n}{n/2} p^{n/2} q^{n-n/2}$ since n even

$$= \frac{n!}{(n-n/2)!\,(n/2)!} (pq)^{n/2}$$

$$= \frac{n!}{((n/2)!)^2} (pq)^{n/2}$$

$$\approx \frac{\sqrt{2\pi n}\, \frac{n^n}{e^n} (p \cdot q)^{n/2}}{\left(\sqrt{2\pi \frac{n}{2}} \left(\frac{n}{2}\right)^{n/2} e^{-n/2}\right)^2}$$ by Stirling's Approximation

$$= \frac{\sqrt{2\pi n}\, \frac{n^n}{e^n} (pq)^{n/2}}{\pi \cdot n \cdot \frac{n^n}{2^n} e^{-n}}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}\, 2^n (pq)^{n/2}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} (4 \cdot p \cdot q)$$

$$\boxed{\therefore p = q = \frac{1}{2} \longrightarrow (4pq)^{n/2} = 1^{n/2} = 1}$$

$\therefore b(n,k,p) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$

QED.

Ex: (computer)

$b(10,5,\frac{1}{2}) = 0.24609$ dbinom(5,10,0.5)

faster decay if $p \neq q$

Stirling: $b(10,5,\frac{1}{2}) \approx 0.25231$ sqrt(2/pi/10)

n	$b(n,\frac{n}{2},\frac{1}{2})$	Stirling's Approx.
100	0.03958	0.03978
1000	0.02522	0.02523
10,000	0.00798	0.00798
100,000	0.00252	0.00252

Compare to: $\frac{\text{\# success}}{\text{\# total}} \longrightarrow p$
(intuition)

Multinomial probability

Trinomial : 3 outcomes "yes", "no", "abstain"

Put $\underline{Z = n - X - Y}$ for n Bernoulli trials

with probabilities p_z, p_x, p_y : $p_x + p_y + p_z = 1$

$$\therefore\ 1 = 1^n = (p_x + p_y + p_z)^n = (\underbrace{p_x}_{a} + \underbrace{(p_y + p_z)}_{b})^n$$

$$\overset{\text{B.T.}}{=} \sum_{x=0}^{n} \binom{n}{x} p_x^{x} \overbrace{(p_y + p_z)^{n-x}}^{(a+b)^n}$$

$$\overset{\text{B.T.}}{=} \sum_{x=0}^{n} \sum_{y=0}^{n-x} \binom{n}{x}\binom{n-x}{y} p_x^{x} p_y^{y} p_z^{n-x-y}$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\,\cancel{(n-x)!}} \cdot \frac{\cancel{(n-x)!}}{y!\,(n-x-y)!} p_x^{x} p_y^{y} p_z^{n-x-y}$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!} p_x^{x} p_y^{y} p_z^{\overbrace{n-x-y}^{z}}$$

$$= \sum_{\substack{x\geq 0, y\geq 0, z\geq 0 \\ x+y+z=n}} \frac{n!}{x!\,y!\,z!} p_x^{x} p_y^{y} p_z^{z}$$

$\therefore$ trinomial pdf: $p(x,y) = P[X = x, Y = y]$

$$= \boxed{\frac{n!}{x!\,y!\,(n-x-y)!} p_x^{x} p_y^{y} p_z^{n-x-y}}$$

$$\frac{n!}{x!\,y!\,(n-x-y)!} = \binom{n}{x,y,z}$$

ways to choose n w/ "x" Type 1, "y" Type 2 and "z" Type 3

Ex: How many ways to assign 20 students to groups A, B, C w/ $k_A = k_B = 6$ and $k_C = 8$

$$\binom{20}{6,6,8} = \frac{20!}{6!\,6!\,8!} = 116{,}396{,}280$$

Note: $\underbrace{\binom{n}{k}}_{\text{binomial}} = \binom{n}{k,\,n-k}$

Ex: Sample w/ replacement

$p_x = 0.30$
$p_y = 0.15$ $\Big\}$ $p_z = 0.55$

① $P[\,2 \text{ "X" and } 3 \text{ "Y" if pick } 10\,]$

$$\binom{10}{2,3,5}(0.30)^2(0.15)^3(0.55)^5$$

② $P[\,2 \text{ "X" if pick } 10\,]$ need "marginal" probability.

$p(x,y)$ describes "joint" probability of X and Y

$p(x)$ and $p(y)$ describe probability of X and Y seperately. "marginal"

Marginal probability of trinomial

$$P(\underbrace{X=x}_{\text{sample } n\text{, prob of } x \text{ "X" regardless "Y" or "Z" count}}) = \underbrace{\sum_{y=0}^{n-x}}_{\text{sum over all "y" w/ "x" fixed}} p(x,y) \overset{\text{tri}}{=} \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!}\, p_x^x\, p_y^y\, p_z^{n-x-y}$$

$$= \frac{(n-x)!}{(n-x)!}\,\frac{n!}{x!}\,p_x^x \cdot \sum_{y=0}^{n-x} \frac{1}{y!\,(n-x-y)!}\, p_y^y\, p_z^{n-x-y}$$

$$= \frac{n!}{x!\,(n-x)!}\,p_x^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!\,(n-x-y)!}\, p_y^y\, p_z^{n-x-y}$$

$$= \binom{n}{x} p_x^x \left[\sum_{y=0}^{n-x} \binom{n-x}{y} p_y^y\, p_z^{(n-x)-y}\right]$$

$$\overset{\text{B.T.}}{=} \binom{n}{x} p_x^x (p_y + p_z)^{n-x} = \binom{n}{x} p_x^x (1 - p_x)^{n-x}$$

$p_x + p_y + p_z = 1$

$\therefore X \sim b(n, x, p_x)$

$$\boxed{\therefore \text{ marginal is binomial}}$$

Conditional probability of trinomial

$$P(X=x \mid Y=y) = p(x|y) = \frac{p(x,y)}{p(y)} = \frac{\frac{\cancel{n!}}{x!\,\cancel{y!}\,(n-x-y)!}\, p_x^x\, \cancel{p_Y^y}\, p_z^{n-x-y}}{\frac{\cancel{n!}}{\cancel{y!}\,(n-y)!}\, \cancel{p_Y^y}\, (1-p_Y)^{n-x-y}}$$

sample n, probability of x "X" given y "Y"

($\therefore n-y-x$ "Z")

$$= \frac{(n-y)!}{x!\,((n-y)-x)!} \cdot \frac{p_x^x \cdot p_z^{(n-y)-x}}{(1-p_Y)^{n-y\underbrace{-x+x}_{=0}}}$$

$$= \binom{n-y}{x}\left(\frac{p_x}{1-p_Y}\right)^x \left(1-\frac{p_x}{1-p_Y}\right)^{(n-y)-x}$$

$$= b\left(n-y,\ x,\ \frac{p_x}{1-p_Y}\right)$$

$\boxed{\therefore \text{ conditional is binomial}}$

Multinomial Theorem:

$$(p_1+\cdots+p_k)^n = \sum_{l_1+\cdots+l_k=n} \frac{n!}{l_1!\,l_2!\cdots l_k!}\, p_1^{l_1}\cdots p_k^{l_k}$$

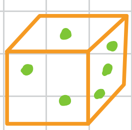
(partition)

if $l_1+\cdots+l_k=n$

Compare: binomial: $p_x,\ p_Y = 1-p_x$

trinomial: $p_x,\ p_Y,\ p_Z = 1-p_x-p_Y$

Ex: - Roll a die ("d6") $\therefore l=6$



- roll fair die 7 times, $\therefore n=7$ $p_1=\cdots=p_6=\frac{1}{6}$

$$\therefore P(3\ 2\text{'s},\ 2\ 4\text{'s},\ 2\ 6\text{'s}) = \binom{7}{0,3,0,2,0,2}\left(\frac{1}{6}\right)^7$$

$$= \frac{7\cdot 6\cdot 5}{6^7} \approx 0.00078$$

- Deep neural classifier w/ K output SoftMax neurons (1-in-K coding)

$\therefore x \longrightarrow N \longrightarrow N(x)$, 1 roll of K-sided die ("categorical" multinomial)

$$\therefore \ln p = \ln p_1^{x_1}\cdots p_K^{x_K} = \sum_{k=1}^{K} x_k \cdot \ln p_k$$ — cross-entropy

$p = p_1^{x_1}\cdots p_K^{x_K}$ unknown   data

data / "observations"

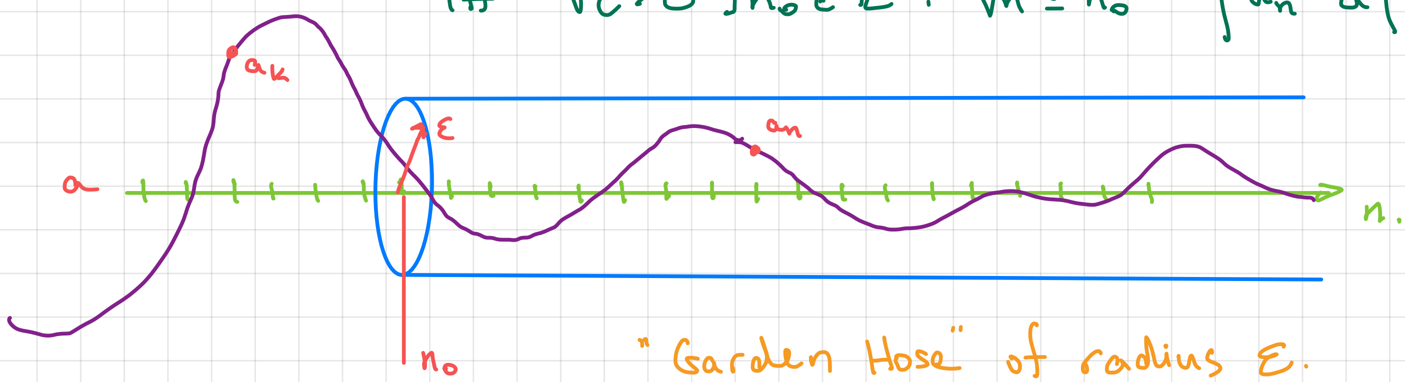
Q: find "best choice" of $p_1,\ldots,p_K$ given $x_1\ldots x_K$

Sequences and series

Defn: Limit of a sequence $a_1, a_2, a_3, \ldots$

$a_n \to a$ iff $\lim_{n\to\infty} a_n = a$

iff $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \quad |a_n - a| < \varepsilon$



"Garden Hose" of radius ε.

$\therefore$ smaller $\varepsilon > 0 \longrightarrow$ Takes longer to find n_0.

Partial Sum $\quad S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N$

Infinite Series $\quad$ " $\sum_{n=0}^{\infty} a_n = S$ " converges

iff $\lim_{N\to\infty} S_N = S$ $\quad$ (S finite)

iff $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 : |S_N - S| < \varepsilon$.

(else "diverges")

Special case:

$\lim_{n\to\infty} a_n = \infty$ iff $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \ a_n > \varepsilon$.

- a type of non-convergence
- not the same in general as $\lim_{n\to\infty} a_n \neq a$.

Thrm: (Triangle Inequality) $|a+b| \le |a| + |b|$

Prf: $|a+b|^2 = (a+b)^2 = a^2 + b^2 + 2ab$

$\le a^2 + b^2 + 2|a|\cdot|b|$ note: $|x| = \max(x,-x)$

$= |a|^2 + |b|^2 + 2|a|\cdot|b|$

$= (|a| + |b|)^2$

$\therefore |a+b| \le |a| + |b|$

Ex: Thrm: $(a_n \longrightarrow a) \wedge (b_n \longrightarrow b) \Longrightarrow (a_n + b_n \longrightarrow a+b)$

Prf: Suppose $a_n \longrightarrow a$ and $b_n \longrightarrow b$

pick $\varepsilon > 0$ $\quad \exists n_0 : \forall n \ge n_0 \quad |a_n - a| < \frac{\varepsilon}{2}$

$\exists m_0 : \forall n \ge m_0 \quad |b_n - b| < \frac{\varepsilon}{2}$

$\therefore \forall n \ge \max(n_0, m_0)$

$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)|$

$\overset{T.I.}{\le} |a_n - a| + |b_n - b| \boxed{< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon}$

QED.

Thrm: $|a-b| \ge \big||a| - |b|\big|$

Prf: $|a| - |b| = |(a-b) + b| - |b| \le |a-b| + \cancel{|b|} - \cancel{|b|} = |a-b|$

$\therefore |a| - |b| \le |a-b|$ similarly: $|b| - |a| \le |a-b|$

$\Downarrow$

$|a| - |b| \ge -|a-b|$

$\therefore \overset{-c}{-|a-b|} \le |a| - |b| \le \overset{c}{|a-b|}$

$\therefore \boxed{\big||a| - |b|\big| \le |a-b|}$

QED.

Fact: $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ where $e^x \triangleq \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Fact: $\sum_{n=0}^{\infty} a_n = s \xrightarrow{\quad} \lim_{n\to\infty} a_n = 0$

$(\not\leftarrow)$

Test: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

- converges if $p > 1$.
- diverges if $p \le 1$

Ex: $\sum_{k=1}^{\infty} \frac{1}{n}$ diverges since $\frac{1}{n} = \frac{1}{n^1}$, $\therefore p = 1$

$\sum_{k=1}^{\infty} \frac{1}{n^2}$ — $n = 2 > 1$

$\sum_{k=1}^{\infty} \frac{1}{n^{1.0001}}$ — $n = 1.0001 > 1$

$\rightarrow$ converge

Test: (Alternating Series test)

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if

$$\begin{cases} - a_k \ge a_{k+1} > 0 \quad \forall k \in \mathbb{Z}^+. \\ - \lim_{n\to\infty} a_n = 0 \end{cases}$$

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges since:

1) $\frac{1}{n} \ge \frac{1}{n+1} \quad \forall n$
2) $\lim_{n\to\infty} \frac{1}{n} = 0.$

Defn: $\sum_{n=0}^{\infty} a_n$ converges absolutely iff $\sum_{n=0}^{\infty} |a_n|$ converges

($\therefore$ treat infinite sums like finite sums)

Facts: (1) Absolute convergence $\longrightarrow$ convergence
$(\not\longleftarrow)$

(2) If
- $\sum_{n=1}^{\infty} a_n$ converges absolutely
- $\sum_{n=1}^{\infty} b_n$ is any rearrangement of $\sum_{n=1}^{\infty} a_n$

Then:
- $\sum_{n=1}^{\infty} b_n$ converges
- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Defn: $\sum_{n=0}^{\infty} a_n$ converges conditionally iff
- $\sum_{n=0}^{\infty} a_n$ converges
- $\sum_{n=0}^{\infty} |a_n|$ diverges

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges conditionally since $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges but $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(extra, not tested)

Thm: (Riemann's Rearrangement Theorem) Pick ANY $x \in \mathbb{R}$. Suppose $\sum_{n=0}^{\infty} a_n$ converges conditionally. Then $\exists$ a re-arrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ where $\sum_{n=0}^{\infty} b_n = x$.
(example: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

☆

Ratio test for convergence of $\sum_{n=0}^{\infty} a_n$ (series)

Put $L \triangleq \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ Then

(1) $L < 1 \implies$ series converges absolutely.

(2) $L > 1 \implies$ series diverges

(3) $L = 1 \implies$ test fails

Ex: Geometric Series : $a_n = a^n$. $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a^n$ $(a \neq 0)$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{a^n}\cdot a}{\cancel{a^n}}\right|$$

$$= \lim_{n\to\infty} |a|.$$

$$= |a|.$$

$\therefore \sum_{n=0}^{\infty} a^n$ converges absolutely if $|a| < 1$.

Ex: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{1}{n+1}\right|$$

$$= |x|\cdot\lim_{n\to\infty}\frac{1}{n+1} = 0$$ $\therefore e^x$ converges $\forall x$.

Ex: $\sum_{n=1}^{\infty}\frac{1}{n}$ $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/n+1}{1/n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$

$\therefore$ test fails.

Fact: $|x| \le c$ iff $-c \le x \le c$ $(c > 0)$

Note: Taylor's series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

a_n

i.e. power series "around 0"

Power Series for variable x

$$\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Ex: Find all x so that $\sum_{n=0}^{\infty} \frac{n}{5^n} x^n$ converges absolutely.

By the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{n \cdot x^n} \cdot \frac{5^n}{5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{5n} \right|$$

$$= \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n} \right)}_{=1} \cdot \frac{1}{5} \cdot |x|.$$

$$= \frac{1}{5}|x|$$

$$< 1$$

iff $|x| < 5$. iff $-5 < x < 5$

1. $x = +5$ $\quad \sum_{n=0}^{\infty} \frac{n}{\cancel{5^n}} \cdot \cancel{5^n} = \sum_{n=0}^{\infty} n$ diverges

2. $x = -5$ $\quad \sum_{n=0}^{\infty} \frac{n}{5^n} \cdot (-5)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n$ diverges

$\therefore \sum_{n=0}^{\infty} \frac{n}{5^n} x^n$ converges iff $-5 < x < 5$ (else diverges)

Thrm: (1) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $c \neq 0$ then it converges absolutely $|x| < |c|$.

(2) If $\sum_{n=0}^{\infty}$ diverges for $d \neq 0$ then it diverges if $|x| > |d|$.

Thrm: Exactly one holds for $\sum_{n=0}^{\infty} a_n x^n$

(1) it converges only if $x = 0$

(2) it converges absolutely for all x (e.g. e^x)

(3) $\exists r > 0$: $\begin{cases} \text{It converges absolutely if } |x| < r. \\ \text{and} \\ \text{It diverges if } |x| > r. \end{cases}$

$\therefore$ Case 3: "radius of convergence" $r > 0$

$(-r, r)$

$X \sim NB$: Negative Binomial

$P(X = n \text{ trials } \underline{\text{until}} \text{ } k \text{ successes})$

$$= \underbrace{\overbrace{P(\text{head on } n^{th} \text{ flip})}^{n^{th} \text{ trial}}}_{\text{independent}} \cdot \underbrace{\overbrace{P(k-1 \text{ heads in first } n-1 \text{ flips})}^{\text{first } n-1 \text{ trial}}}_{\text{independent}}$$

$$= p \cdot \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{(n-1)-(k-1)}$$

$$= \boxed{\binom{n-1}{k-1} p^{k} (1-p)^{n-k}}$$

$X \sim G(p)$: Geometric

$P(X = n \text{ trials } \underline{\text{until}} \text{ } 1^{st} \text{ success})$

$$\therefore \quad = NB(n, k=1)$$

$$= \binom{n-1}{0} p^{1} (1-p)^{n-1}$$

$$= \boxed{p \cdot (1-p)^{n-1}}$$

Thrm: $$\sum_{k=0}^{n} a^k = \begin{cases} n+1 & \text{if } a=1 \\ \dfrac{1-a^{n+1}}{1-a} & \text{if } a\neq 1. \end{cases}$$

Prf: $S \overset{\Delta}{=} a^0 + a^1 + \cdots + a^n = \sum_{k=0}^{n} a^k$

$\therefore aS = a + a^2 + \cdots + a^{n+1}$

$\therefore S(1-a) = S - a\cdot S$

$= (1+a+\cdots+a^n) - (a+a^2+\cdots+a^{n+1})$

$= 1-a^{n+1}$

$\therefore S = \dfrac{1-a^{n+1}}{1-a}$ since $a \neq 1$

QED

Corr 1: $\sum_{k=0}^{\infty} a^k = \dfrac{1}{1-a}$ if $|a|<1$

Corr 2: $\sum_{k=1}^{\infty} a^k = \dfrac{a}{1-a}$ if $|a|<1$

Corr 3: $\sum_{k=j}^{\infty} a^k = \dfrac{a^j}{1-a}$ if $|a|<1$

Q: verify geometric is a "valid" probability

$$\sum_{\Omega} p(x) \overset{G(p)}{=} \sum_{n=1}^{\infty} p\cdot(1-p)^{n-1} = p\cdot \sum_{n=1}^{\infty}(1-p)^{n-1}$$

Let $k = n-1$
$\therefore n = k+1$

$$= p\cdot \sum_{k+1=1}^{k+1=\infty} (1-p)^k$$

$$= p \sum_{k=0}^{\infty} (\underbrace{1-p}_{0\,\leq\, 1-p\,<\,1})^k$$

$$= p\cdot \frac{1}{1-(1-p)}$$

$$= p\cdot \frac{1}{p} = 1$$

$\therefore$ "valid" probability

Thrm: $P(X > k) = q^k$ if $X \sim G(p)$

$\therefore P(X \leq k) = 1 - q^k$

Prf: $P(X > k) = P(\{X = k+1\} \underbrace{\cup}_{\text{"OR"}} \{X = k+2\} \underbrace{\cup}_{\text{"OR"}} \cdots)$

$$\overset{\text{CAT}}{=} \sum_{x=k+1}^{\infty} p(x)$$

$$\overset{X\sim G(p)}{=} \sum_{x=k+1}^{\infty} p \cdot q^{x-1} \overset{\substack{y = x-1 \\ \therefore x = y+1}}{=} p \cdot \sum_{y=k}^{\infty} q^y$$

$$\overset{\text{Corr 3}}{=} p \cdot \frac{q^k}{1-q} = \frac{p}{p} \cdot q^k = q^k$$

QED.

Fact: $\binom{N}{n} = \sum_{k=0}^{n} \binom{N_1}{k}\binom{N_2}{n-k}$ if $\underbrace{N}_{\text{total pop.}} = \underbrace{N_1}_{\text{\# type 1}} + \underbrace{N_2}_{\text{\# type 2}}$ and $\binom{k}{j} = 0$ for $j > k$.

sum to N

sum to n

Hypergeometric probability

Suppose batch of N items. Sample n w/o replacement.

N_1 type 1 $\quad$ N_2 type 2

$$\binom{N}{n} = \binom{N_1 + N_2}{n} = \underbrace{\sum_{k=0}^{n}}_{\substack{\text{add over all ways for} \\ 0 \text{ "good"}, 1 \text{ "good"}, \cdots n \text{ "good"}}} \underbrace{\binom{N_1}{k}\binom{N_2}{n-k}}_{\substack{\text{\# different ways to draw } k \text{ "good" (Type 1)} \\ \text{and } n-k \text{ "bad" (Type 2)}}}$$

$$\frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}} \geq 0 \quad \text{and} \quad \sum_{k=0}^{n} \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}} = 1.$$

$\therefore \underbrace{\dfrac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}}_{\text{hypergeometric p.m.f.}}$ obeys all axioms of probability.

w/ $P[k \text{ "good" from draw of } n]$

Ex: small lake, $N = 50$ fish $\rightarrow$ $N_1 = 10$ trout

$\searrow$ $N_2 = 40$ large mouth bass

Bob catches: 7 fish w/ replacement ("catch and release")

$$P[\text{2 trout and 5 bass}] = b(7, 2, \tfrac{1}{5})$$
$$= \binom{7}{2}\left(\tfrac{1}{5}\right)^2\left(\tfrac{4}{5}\right)^5$$
$$= 0.275$$

7 fish w/o replacement ("catch and eat")

$$P[\text{2 trout and 5 bass}] = \text{Hypergeo}(\overset{n}{7}, \overset{N_1}{10}, \overset{k}{2}, \overset{N}{50})$$
$$= \frac{\binom{10}{2}\binom{40}{5}}{\binom{50}{7}}$$
$$= 0.2969$$

Q: probability w/ replacement and w/o replacement are similar. Under what conditions is this a "good" approximation?
(i.e. when can you act as if "with replacement")

A: if the pond is an ocean. $\longrightarrow$ N big, n small
(fixed ratio $\frac{N_1}{N}$)

Binomial approximation to the hypergeometric

Thrm Hypergeometric $\xrightarrow{d}$ Binomial if $N \to \infty$ and $p = \frac{n_1}{N}$

Prf: Say $X \sim Hyp(n, m, k, N)$. Fixed k & n

$$\therefore \lim_{N\to\infty} P(X=k) = \lim_{N\to\infty} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{m!}{k!\,(m-k)!} \cdot \frac{(N-n)!\,n!}{N!} \cdot \frac{(N-m)!}{(n-k)!\,(N-n-m+k)!}$$

$$= \lim_{N\to\infty} \frac{\frac{n!}{(n-k)!\,k!} \cdot \frac{m!\,(N-m)!}{(N-m-n+k)!\,(m-k)!}}{\frac{N!}{(N-n)!}}$$

$$= \lim_{N\to\infty} \binom{n}{k} \frac{\frac{m!}{(m-k)!} \cdot \frac{(N-m)!}{(N-m-n+k)!}}{\frac{N!}{(N-n)!}}$$

$$= \lim_{N\to\infty} \frac{\prod_{j=1}^{k}(m-k+j)\prod_{j=1}^{N-k}(N-m-n+k+j)}{\prod_{j=1}^{n}(N-n+j)} \cdot \frac{N^n}{N^n}$$

$\binom{n}{k}$ fixed

$$= \binom{n}{k} \cdot \lim_{N\to\infty} \frac{\prod_{j=1}^{k}\left(\overbrace{\frac{m}{N}}^{p} - \cancel{\frac{k}{N}} + \cancel{\frac{j}{N}}\right)\prod_{j=1}^{n-k}\left(\overbrace{1-\frac{m}{N}}^{1-p} - \cancel{\frac{n}{N}} + \cancel{\frac{k}{N}} + \cancel{\frac{j}{N}}\right)}{\prod_{j=1}^{n}\left(1 - \cancel{\frac{n}{N}} + \cancel{\frac{j}{N}}\right)}$$

$$= \binom{n}{k} \cdot \left(\frac{m}{N}\right)^k \left(1-\frac{m}{N}\right)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= b(n, k, p)$$

QED

BEG CUP

Sampling

Outcomes	w/ replacement	w/o replacement
2	Binomial negative binomial geometric "until"	Hypergeometric
≥ 2	Multinomial	Multivariate Hypergeometric

Hypergeo → Binomial

Multi Hypergeo → Multinomial

$$\frac{\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_k}{n_k}}{\binom{N}{n}}$$

$n = n_1 + \cdots + n_k$

$N = N_1 + \cdots + N_k$

Issue-Spotting Sequence

Q1: Outcomes? (2 or more?)

Q2: Sampling? (w/ or w/o replacement)

Q3: Until structure?

"UNTIL" ⟹ negative binomial (k=1 outcomes, geometric)

UNTIL

- Discrete → Negative Binomial (k=1: Geometric)
- Continuous → γ (gamma) (k=1: Exponential; k>1: Erlang $(k \in \mathbb{Z}^+)$)